Problem Set: Fiscal Policy, Government Debt, and Capital Crowding Out

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers that only state final results receive limited credit. Unless otherwise stated, all variables are positive and all rates are expressed in decimal form.

Question 1: Fiscal Policy, Debt Dynamics, and Intertemporal Solvency [Total: 50 marks]

During recessions and wars, governments often run large deficits. During expansions, they may be expected to run surpluses to stabilise public debt. This question studies debt accumulation, debt-to-GDP dynamics, and the intertemporal government budget constraint.

Let D_t denote the nominal stock of public debt at the end of period t, i_t the nominal interest rate, G_t government purchases, Tr_t transfers, T_t tax revenue, P_tY_t nominal GDP, g_t real GDP growth, and π_t inflation. Suppose money finance is unavailable.

(a) Starting from the flow government budget constraint, derive

$$D_t = (1+i_t)D_{t-1} + G_t + Tr_t - T_t.$$

Define the primary balance as

$$PB_t \equiv T_t - G_t - Tr_t.$$

Then derive the exact debt-to-GDP law of motion:

$$d_t = \frac{1+i_t}{(1+g_t)(1+\pi_t)} d_{t-1} - pb_t,$$

where

$$d_t \equiv \frac{D_t}{P_tY_t}, \qquad pb_t \equiv \frac{PB_t}{P_tY_t}.$$

Finally derive the approximate law

$$\Delta d_t \approx (r_t - g_t)d_{t-1} - pb_t,$$

where r_t is the real interest rate. **[12 marks]**

(b) Suppose r and g are constant and define

$$R \equiv \frac{1+r}{1+g}.$$

Assume the primary balance ratio is constant at $pb_t = p$. Solve the first-order difference equation

$$d_t = Rd_{t-1} - p.$$

Derive the primary balance required to stabilise the debt ratio at a target $\bar{d}$. Explain the cases $r > g$, $r = g$, and $r < g$. **[10 marks]**

(c) Consider a country with initial debt ratio

$$d_0 = 0.90,$$

nominal interest rate

$$i = 0.05,$$

inflation

$$\pi = 0.02,$$

real GDP growth

$$g = 0.015,$$

and a primary deficit equal to 2% of GDP, so that

$$pb = -0.02.$$

Using the exact debt-to-GDP equation, compute d_1, d_2, d_3, d_4, d_5. Interpret the fiscal sustainability implication. **[10 marks]**

(d) Consider a two-period economy with no money finance and no transfers. The government begins period 1 with debt D_1. Derive the two-period intertemporal government budget constraint:

$$T_1 - G_1 + \frac{T_2 - G_2}{1+i} = (1+i)D_1.$$

Suppose

$$i = 0.04, \qquad D_1 = 0.30Y, \qquad G_1 = 0.22Y, \qquad G_2 = 0.20Y, \qquad T_1 = 0.18Y.$$

Compute the required value of T_2/Y if the government must end period 2 with zero debt. **[10 marks]**

(e) Starting from the national income identity for an open economy,

$$Y = C + I + G + EX - IM,$$

derive

$$I = (Y - T - C) + (T - G) + (IM - EX).$$

Use this identity to explain analytically why a larger government deficit may crowd out investment, and state two mechanisms that may offset full crowding out. **[8 marks]**

Question 2: Government Debt, Saving, and Ricardian Equivalence in an OLG Economy [Total: 50 marks]

Consider a two-period-lived overlapping-generations economy. Population is constant. Each individual receives y_1 goods when young and y_2 goods when old. There is no money. Individuals save by holding physical capital or one-period government bonds. Both assets earn the same gross return $1+r$. The young in period t pay a lump-sum tax $\tau_{1,t}$, and the old in period $t+1$ pay $\tau_{2,t+1}$.

The individual budget constraints are

$$c_{1,t} + s_t = y_1 - \tau_{1,t},$$
$$c_{2,t+1} = y_2 + (1+r)s_t - \tau_{2,t+1}.$$

Government bonds per young person are b_t, physical capital per young person is k_t, and total private saving satisfies

$$s_t = k_t + b_t.$$

(a) Suppose preferences are

$$U(c_{1,t}, c_{2,t+1}) = \ln c_{1,t} + \beta \ln c_{2,t+1}, \qquad \beta > 0.$$

Derive the lifetime budget constraint, the Euler equation, optimal $c_{1,t}$ and $c_{2,t+1}$, and the saving function

$$s_t = s(y_1, y_2, \tau_{1,t}, \tau_{2,t+1}, r, \beta).$$

Then derive the signs of

$$\frac{\partial s_t}{\partial \tau_{1,t}} \quad \text{and} \quad \frac{\partial s_t}{\partial \tau_{2,t+1}}.$$

[14 marks]

(b) The government cuts taxes on the young in period t by $x > 0$:

$$\tau_{1,t}^N = \tau_{1,t} - x,$$

with no change in current government expenditure. In Case 1, the extra debt is rolled over and is eventually repaid by future generations, not by generation t. Derive

$$\Delta b_t = x, \qquad \Delta b_{t+j} = x(1+r)^j \quad \text{for } j \geq 1$$

before final retirement. Under log utility, compute

$$\Delta c_{1,t}, \qquad \Delta c_{2,t+1}, \qquad \Delta s_t, \qquad \Delta k_t.$$

Show that government debt crowds out capital. **[12 marks]**

(c) In Case 2, the same tax cut x for the young in period t is repaid by taxing the same generation when old in period $t+1$. Derive the required tax increase

$$\Delta \tau_{2,t+1} = (1+r)x.$$

Then prove that lifetime wealth, consumption, and capital are unchanged:

$$\Delta c_{1,t} = 0, \qquad \Delta c_{2,t+1} = 0, \qquad \Delta k_t = 0.$$

Explain why private saving rises one-for-one with government debt. **[10 marks]**

(d) Now drop log utility. Suppose only that both $c_{1,t}$ and $c_{2,t+1}$ are normal goods. Prove that in Case 1,

$$0 < \Delta s_t < x$$

and therefore

$$\Delta k_t < 0.$$

Then explain why in Case 2 Ricardian equivalence holds even without log utility. **[8 marks]**

(e) Suppose households internalise only a fraction $\mu \in [0,1]$ of the future tax burden generated by the tax cut. Under log utility, the effective present-value wealth gain is

$$(1-\mu)x.$$

Derive

$$\Delta s_t = \frac{\beta + \mu}{1+\beta} x$$

and

$$\Delta k_t = -\frac{1-\mu}{1+\beta} x.$$

Interpret the two polar cases $\mu = 0$ and $\mu = 1$. **[6 marks]**

Detailed Solutions

Solution to Question 1 [50 marks]

Part (a) [12 marks]

With no money finance, the flow government budget constraint is

$$G_t + Tr_t + i_t D_{t-1} = T_t + (D_t - D_{t-1}).$$

Rearrange:

$$D_t - D_{t-1} = G_t + Tr_t + i_t D_{t-1} - T_t.$$

Therefore,

$$D_t = (1+i_t)D_{t-1} + G_t + Tr_t - T_t.$$

Define the primary balance:

$$PB_t \equiv T_t - G_t - Tr_t.$$

Then

$$\boxed{D_t = (1+i_t)D_{t-1} - PB_t.}$$

Divide by nominal GDP P_tY_t:

$$\frac{D_t}{P_tY_t} = (1+i_t)\frac{D_{t-1}}{P_tY_t} - \frac{PB_t}{P_tY_t}.$$

Since

$$P_tY_t = (1+\pi_t)(1+g_t)P_{t-1}Y_{t-1},$$

we have

$$\frac{D_{t-1}}{P_tY_t} = \frac{D_{t-1}}{P_{t-1}Y_{t-1}}\frac{1}{(1+\pi_t)(1+g_t)}.$$

Thus,

$$\boxed{d_t = \frac{1+i_t}{(1+g_t)(1+\pi_t)}d_{t-1} - pb_t.}$$

Using the Fisher approximation

$$i_t \approx r_t + \pi_t,$$

we obtain

$$\frac{1+i_t}{(1+g_t)(1+\pi_t)} \approx \frac{1+r_t}{1+g_t} \approx 1 + r_t - g_t.$$

Therefore,

$$d_t \approx (1+r_t-g_t)d_{t-1} - pb_t.$$

Subtracting d_{t-1} from both sides gives

$$\boxed{\Delta d_t \approx (r_t - g_t)d_{t-1} - pb_t.}$$

The term $(r_t - g_t)d_{t-1}$ is the automatic debt-dynamics component. When the real interest rate exceeds real growth, existing debt tends to grow relative to GDP unless the government runs a sufficiently large primary surplus.

Marking guide: flow debt equation, 3; primary balance definition, 2; exact debt-ratio equation, 4; approximation and interpretation, 3.

Part (b) [10 marks]

The debt-ratio equation is

$$d_t = Rd_{t-1} - p, \qquad R = \frac{1+r}{1+g}.$$

Iterating forward gives

$$d_1 = Rd_0 - p,$$
$$d_2 = R^2 d_0 - p(1+R),$$

and therefore

$$\boxed{d_t = R^t d_0 - p\sum_{j=0}^{t-1} R^j.}$$

If $R \neq 1$,

$$\sum_{j=0}^{t-1} R^j = \frac{1-R^t}{1-R}.$$

Hence,

$$\boxed{d_t = R^t d_0 - p\frac{1-R^t}{1-R}.}$$

If $R = 1$,

$$\boxed{d_t = d_0 - tp.}$$

To stabilise debt at $\bar{d}$, impose

$$d_t = d_{t-1} = \bar{d}.$$

Then

$$\bar{d} = R\bar{d} - p.$$

Therefore,

$$\boxed{p = (R-1)\bar{d}.}$$

Since

$$R - 1 = \frac{1+r}{1+g} - 1 = \frac{r-g}{1+g},$$

the stabilising primary balance is

$$\boxed{p^{stab} = \frac{r-g}{1+g}\bar{d}.}$$

If $r > g$, then $p^{stab} > 0$: the government needs a primary surplus to stabilise debt. If $r = g$, then $p^{stab} = 0$: a balanced primary budget stabilises debt. If $r < g$, then $p^{stab} < 0$: the government can run a primary deficit while keeping the debt ratio constant.
Marking guide: solution of difference equation, 4; stabilising primary balance, 3; interpretation of $r - g$, 3.

Part (c) **[10 marks]**

The exact gross debt-dynamics factor is

$$R_N = \frac{1+i}{(1+g)(1+\pi)} = \frac{1.05}{(1.015)(1.02)}.$$

Since

$$(1.015)(1.02) = 1.0353,$$
$$R_N = \frac{1.05}{1.0353} \approx 1.0142.$$

The debt equation is

$$d_t = 1.0142d_{t-1} - (-0.02) = 1.0142d_{t-1} + 0.02.$$

Starting from $d_0 = 0.90$:

$$d_1 = 1.0142(0.90) + 0.02 \approx 0.9328,$$
$$d_2 = 1.0142(0.9328) + 0.02 \approx 0.9660,$$
$$d_3 = 1.0142(0.9660) + 0.02 \approx 0.9997,$$
$$d_4 = 1.0142(0.9997) + 0.02 \approx 1.0339,$$
$$d_5 = 1.0142(1.0339) + 0.02 \approx 1.0686.$$

Thus:

t	1	2	3	4	5
d_t	0.9328	0.9660	0.9997	1.0339	1.0686

The debt ratio rises from 90% to approximately 106.9% of GDP in five periods. The reason is twofold: the nominal interest factor exceeds nominal GDP growth, and the government also runs a primary deficit. Stabilisation would require replacing the primary deficit with a primary surplus.
Marking guide: exact growth factor, 2; recursive calculations, 5; final table, 1; sustainability interpretation, 2.

Part (d) **[10 marks]**

With no transfers and no money finance, the government budget constraint is

$$D_{t+1} = (1+i)D_t + G_t - T_t.$$

For period 1:

$$D_2 = (1+i)D_1 + G_1 - T_1.$$

For period 2, impose zero terminal debt:

$$0 = (1+i)D_2 + G_2 - T_2.$$

Thus,

$$T_2 - G_2 = (1+i)D_2.$$

Substitute D_2:

$$T_2 - G_2 = (1+i)\left[(1+i)D_1 + G_1 - T_1\right].$$

Divide both sides by $(1+i)$:

$$\frac{T_2 - G_2}{1+i} = (1+i)D_1 + G_1 - T_1.$$

Rearranging gives

$$\boxed{T_1 - G_1 + \frac{T_2 - G_2}{1+i} = (1+i)D_1.}$$

Now substitute the numerical values:

$$0.18Y - 0.22Y + \frac{T_2 - 0.20Y}{1.04} = 1.04(0.30Y).$$

Thus,

$$-0.04Y + \frac{T_2 - 0.20Y}{1.04} = 0.312Y.$$

Therefore,

$$\frac{T_2 - 0.20Y}{1.04} = 0.352Y.$$

Hence,

$$T_2 - 0.20Y = 1.04(0.352Y) = 0.36608Y.$$

So

$$\boxed{\frac{T_2}{Y} = 0.56608.}$$

The required second-period tax revenue is approximately 56.6% of GDP. This illustrates that budgets need not balance period by period, but they must balance in present value.
Marking guide: two-period derivation, 5; numerical computation, 4; interpretation, 1.

Part (e) **[8 marks]**

Start from

$$Y = C + I + G + EX - IM.$$

Rearrange:

$$I = Y - C - G - EX + IM.$$

Add and subtract taxes T:

$$I = (Y - T - C) + (T - G) + (IM - EX).$$

Thus,

$$\boxed{I = (Y - T - C) + (T - G) + (IM - EX).}$$

The term

$$Y - T - C$$

is private saving. The term

$$T - G$$

is government saving. The term

$$IM - EX$$

is foreign saving, or the current-account deficit.

A larger deficit means $T - G$ falls. Holding private saving and foreign saving fixed, investment must fall one-for-one:

$$\Delta I = \Delta(T - G) < 0.$$

This is crowding out.

However, full crowding out may be offset if private saving rises or if foreign saving increases. Private saving may rise under Ricardian equivalence if households anticipate future taxes. Foreign saving may increase if higher domestic interest rates attract capital inflows from abroad.

Marking guide: identity derivation, 3; crowding-out logic, 3; offsetting mechanisms, 2.

Solution to Question 2 [50 marks]

Part (a) [14 marks]

The two budget constraints are

$$c_{1,t} + s_t = y_1 - \tau_{1,t},$$

and

$$c_{2,t+1} = y_2 + (1+r)s_t - \tau_{2,t+1}.$$

Solve the first constraint for saving:

$$s_t = y_1 - \tau_{1,t} - c_{1,t}.$$

Substitute into the second constraint:

$$c_{2,t+1} = y_2 + (1+r)(y_1 - \tau_{1,t} - c_{1,t}) - \tau_{2,t+1}.$$

Rearrange:

$$c_{1,t} + \frac{c_{2,t+1}}{1+r} = y_1 - \tau_{1,t} + \frac{y_2 - \tau_{2,t+1}}{1+r}.$$

Define lifetime wealth:

$$\boxed{w_t = y_1 - \tau_{1,t} + \frac{y_2 - \tau_{2,t+1}}{1+r}.}$$

Then the lifetime budget constraint is

$$\boxed{c_{1,t} + \frac{c_{2,t+1}}{1+r} = w_t.}$$

The household solves

$$\max_{c_{1,t}, c_{2,t+1}} \ln c_{1,t} + \beta \ln c_{2,t+1}$$

subject to the lifetime budget constraint. The Euler equation is

$$\frac{1/c_{1,t}}{\beta / c_{2,t+1}} = 1 + r.$$

Hence,

$$\boxed{c_{2,t+1} = \beta(1+r)c_{1,t}.}$$

Substitute into the lifetime budget constraint:

$$c_{1,t} + \frac{\beta(1+r)c_{1,t}}{1+r} = w_t.$$

Thus,

$$(1+\beta)c_{1,t} = w_t.$$

Therefore,

$$\boxed{c_{1,t} = \frac{w_t}{1+\beta},}$$

and

$$\boxed{c_{2,t+1} = \frac{\beta(1+r)w_t}{1+\beta}.}$$

Saving is

$$s_t = y_1 - \tau_{1,t} - c_{1,t}.$$

Thus,

$$s_t = y_1 - \tau_{1,t} - \frac{1}{1+\beta}\left[y_1 - \tau_{1,t} + \frac{y_2 - \tau_{2,t+1}}{1+r}\right].$$

Simplify:

$$\boxed{s_t = \frac{\beta}{1+\beta}(y_1 - \tau_{1,t}) - \frac{1}{(1+\beta)(1+r)}(y_2 - \tau_{2,t+1}).}$$

Therefore,

$$\boxed{\frac{\partial s_t}{\partial \tau_{1,t}} = -\frac{\beta}{1+\beta} < 0,}$$

and

$$\boxed{\frac{\partial s_t}{\partial \tau_{2,t+1}} = \frac{1}{(1+\beta)(1+r)} > 0.}$$

A higher tax when young reduces disposable income and saving. A higher expected tax when old raises saving because the household saves more to finance old-age consumption.

Marking guide: lifetime budget constraint, 3; Euler equation, 3; optimal consumption, 3; saving function, 3; comparative statics, 2.

Part (b) **[12 marks]**

The government cuts taxes on the young by x:

$$\tau_{1,t}^N = \tau_{1,t} - x.$$

With no change in government expenditure, the period-t deficit rises by x, so

$$\boxed{\Delta b_t = x.}$$

If the debt is rolled over and taxes are not raised before the final retirement date, then next period the government must issue new debt to cover principal plus interest:

$$\Delta b_{t+1} = (1+r)\Delta b_t = (1+r)x.$$

Repeating the argument:

$$\boxed{\Delta b_{t+j} = x(1+r)^j, \qquad j \geq 1.}$$

Because generation t receives a tax cut and does not pay the later tax increase, its lifetime wealth rises by

$$\Delta w_t = x.$$

Under log utility,

$$c_{1,t} = \frac{w_t}{1+\beta}.$$

Therefore,

$$\boxed{\Delta c_{1,t} = \frac{x}{1+\beta}.}$$

Also,

$$c_{2,t+1} = \frac{\beta(1+r)w_t}{1+\beta}.$$

Thus,

$$\boxed{\Delta c_{2,t+1} = \frac{\beta(1+r)x}{1+\beta}.}$$

Saving satisfies

$$s_t = y_1 - \tau_{1,t} - c_{1,t}.$$

The tax cut raises disposable young income by x, so

$$\Delta s_t = x - \Delta c_{1,t}.$$

Hence,

$$\Delta s_t = x - \frac{x}{1+\beta} = \boxed{\frac{\beta}{1+\beta}x.}$$

Since

$$s_t = k_t + b_t,$$

we have

$$\Delta s_t = \Delta k_t + \Delta b_t.$$

Therefore,

$$\Delta k_t = \Delta s_t - \Delta b_t = \frac{\beta}{1+\beta}x - x.$$

Thus,

$$\boxed{\Delta k_t = -\frac{x}{1+\beta} < 0.}$$

Government debt crowds out capital because private saving rises by less than the increase in government bonds.
Marking guide: debt issuance and rollover path, 4; consumption responses, 3; saving response, 2; capital crowding out, 3.

Part (c) **[10 marks]**

In Case 2, the same generation that receives the tax cut when young pays the tax increase when old. The government must collect enough old-age tax revenue to repay the extra debt plus interest:

$$\boxed{\Delta\tau_{2,t+1} = (1+r)x.}$$

The change in lifetime wealth is

$$\Delta w_t = x - \frac{(1+r)x}{1+r}.$$

Therefore,

$$\boxed{\Delta w_t = 0.}$$

Since preferences and relative prices are unchanged, optimal lifetime consumption is unchanged:

$$\boxed{\Delta c_{1,t} = 0, \qquad \Delta c_{2,t+1} = 0.}$$

However, the tax cut raises disposable income when young by x. Because first-period consumption is unchanged,

$$\Delta s_t = x - \Delta c_{1,t} = x.$$

Thus,

$$\boxed{\Delta s_t = x.}$$

The government issues exactly x more bonds:

$$\Delta b_t = x.$$

Since

$$\Delta k_t = \Delta s_t - \Delta b_t,$$

we obtain

$$\boxed{\Delta k_t = x - x = 0.}$$

Private saving rises one-for-one with government debt because households save the entire tax cut in anticipation of the old-age tax increase required to retire the debt. Hence government debt is neutral.
Marking guide: required old-age tax increase, 3; unchanged lifetime wealth, 2; unchanged consumption, 2; saving and capital effects, 3.

Part (d) **[8 marks]**

In Case 1, generation t receives a tax cut of size x and does not pay the future tax increase. Its lifetime wealth rises. Since both $c_{1,t}$ and $c_{2,t+1}$ are normal goods,

$$\Delta c_{1,t} > 0, \qquad \Delta c_{2,t+1} > 0.$$

The young-period budget constraint implies

$$s_t = y_1 - \tau_{1,t} - c_{1,t}.$$

After the tax cut,

$$s_t^N = y_1 - (\tau_{1,t} - x) - c_{1,t}^N.$$

Subtracting the original saving equation:

$$\Delta s_t = x - \Delta c_{1,t}.$$

Since $\Delta c_{1,t} > 0$,

$$\Delta s_t < x.$$

The old-age budget constraint is

$$c_{2,t+1} = y_2 + (1+r)s_t - \tau_{2,t+1}.$$

In Case 1, $\tau_{2,t+1}$ is unchanged for generation t. Since $\Delta c_{2,t+1} > 0$,

$$(1+r)\Delta s_t > 0.$$

Thus,

$$\Delta s_t > 0.$$

Combining both inequalities:

$$\boxed{0 < \Delta s_t < x.}$$

Since the government issues x more bonds,

$$\Delta b_t = x.$$

Therefore,

$$\Delta k_t = \Delta s_t - \Delta b_t < 0.$$

Hence,

$$\boxed{\Delta k_t < 0.}$$

In Case 2, the tax cut and the future tax increase apply to the same generation and have exactly offsetting present values. Lifetime wealth is unchanged. Since preferences and prices are unchanged, the same consumption bundle remains optimal. Therefore first-period consumption is unchanged, the entire tax cut is saved, and capital remains unchanged. This argument does not depend on log utility.

Marking guide: normal-goods proof of $0 < \Delta s_t < x$, 5; proof of $\Delta k_t < 0$, 1; Ricardian-equivalence explanation, 2.

Part (e) **[6 marks]**

Suppose households internalise a fraction μ of the future tax burden. The effective wealth gain is

$$\Delta w_t = (1-\mu)x.$$

Under log utility,

$$c_{1,t} = \frac{w_t}{1+\beta}.$$

Thus,

$$\Delta c_{1,t} = \frac{(1-\mu)x}{1+\beta}.$$

Saving changes according to

$$\Delta s_t = x - \Delta c_{1,t},$$

because the young-period tax cut raises disposable income by x. Hence,

$$\Delta s_t = x - \frac{(1-\mu)x}{1+\beta}.$$

Therefore,

$$\boxed{\Delta s_t = \frac{\beta+\mu}{1+\beta}x.}$$

Since

$$\Delta b_t = x,$$

capital changes by

$$\Delta k_t = \Delta s_t - \Delta b_t.$$

Thus,

$$\Delta k_t = \frac{\beta+\mu}{1+\beta}x - x.$$

Therefore,

$$\boxed{\Delta k_t = -\frac{1-\mu}{1+\beta}x.}$$

If $\mu = 0$, households ignore the future tax burden. Then

$$\Delta k_t = -\frac{x}{1+\beta} < 0,$$

which is full crowding out in the log-utility case. If $\mu = 1$, households fully internalise the future tax burden. Then

$$\Delta k_t = 0,$$

which is Ricardian equivalence.

Marking guide: effective wealth change, 1; saving response, 2; capital response, 2; interpretation of polar cases, 1.